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INTERFACE CRACK IN BI-PIEZOTHERMOELASTIC MEDIA AND THE INTERACTION WITH A POINT HEAT SOURCE

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Abstract—Using the extended version of Eshelby–Stroh's formulation and the method of analytical continuation, the problems of interface cracks are reduced to a Hilbert problem of vector form. A general explicit closed form solution for piezothermoelastic interface crack problem is then obtained, the whole field solutions of temperature, heat flux, displacements, electric field, stress and electric induction are given, the explicit expressions for the crack opening displacements and electric potential are also provided. A solution is obtained for the interaction problem between the interface cracks and a point heat source. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Due to the intrinsic thermo-electro-mechanical coupling behavior, piezothermoelastic materials have been widely used in engineering as sensors and transducers, and more recently as actuators. In order to insure the structural integrity of thermo-electro-mechanical devices using these materials, understanding of fracture behavior of piezothermoelastic materials is of great importance. But thermo-electro-mechanical modeling of piezothermoelastic materials is complicated by the fact that the piezothermoelastic materials exhibit thermo-electro-elastic coupling behavior as well as anisotropic behavior.

The fracture analysis of interface cracks has been studied by many researchers, such as Comninou (1977) and Dundurs (1968), who had investigated the physically objectionable behavior of interface crack. Lekhnitskii (1963) and Eshelby *et al.* (1953), Stroh (1958) developed a general method of analyzing an anisotropic elastic material on the basis of stress-function and displacements, respectively. Suo (1990) made an important contribution to solving anisotropic interface cracks. Based on these methods for anisotropic material, the methods to treat piezoelectric materials are developed. Deeg (1980) and Pak (1992) analyzed piezoelectric cracks by generalizing a distributed dislocations method, Sosa (1992) obtained a general solution to a transversely isotropic piezoelectric problem. Suo *et al.* (1992) derived a general solution to the problem of the interface cracks in a bi-piezoelectric material. Kuo and Barnett (1991) carried out an asymptotic crack tip analysis, and found a new type of singularity for an interface crack between bonded piezoelectrics. The steady state thermoelastic problems of anisotropic media were studied by Clements (1973) and Atkinson and Clements (1977). As for the thermoelastic interface crack problem in dissimilar anisotropic media, solutions were presented by Hwu (1992). In the area of thermoelasticity, the point heat source problem deserves special attention, since solutions for general loading can be constructed straightforwardly from this kind of solution. However, up to now, only the singular thermoelastic solutions for an infinite isotropic homogeneous medium having a crack under the action of a point heat source was studied by Zhang and Hasebe (1993).

In this paper, we treat the piezothermoelastic materials which often appear in engineering. We developed a general solution for crack problems subjected to a uniform heat flux and loading on the crack surfaces in a homogeneous or bi-piezothermoelastic media.

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We also present singular solutions for the interaction of a point heat source with the interface crack.

2. BASIC EQUATIONS AND GENERAL SOLUTION

In a fixed rectangular coordinate system ox_i $(i = 1, 2, 3)$, let $\mathbf{u}$, φ, $\boldsymbol{\sigma}$, $\mathbf{D}$, $\mathbf{E}$, T and h be the displacements, electric potential, stress, electric induction, electric field, temperature and heat flow, respectively. The complete set of governing equations for piezothermoelastic problem are (see e.g. Suo *et al.*, 1992; Nowacki, 1986; Kuo and Barnett, 1991).

$$\sigma_{ij} = c_{ijkl}\gamma_{kl} - e_{ijs}E_s - \beta_{ij}T \tag{2.1}$$

$$D_i = \varepsilon_{is}E_s + e_{irs}\gamma_{rs} + \tau_i T \tag{2.2}$$

$$h_i = -\lambda_{ij}T_{,j} \tag{2.3}$$

$$\gamma_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.4}$$

$$E_i = -\varphi_{,i} \tag{2.5}$$

$$\sigma_{ij,i} = 0 \tag{2.6}$$

$$D_{i,i} = 0 \tag{2.7}$$

$$h_{i,i} = 0 \tag{2.8}$$

where

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}; \quad e_{kij} = e_{kji}; \quad e_{ij} = e_{kji}; \quad \varepsilon_{ij} = \varepsilon_{ji}; \quad \beta_{ij} = \beta_{ji} \tag{2.9}$$

with c_{ijkl}, e_{kij}, ε_{ij}, β_{ij}, τ_i, λ_{ij} are the elasticity constants, piezoelectricity constants, permittivity constants, stress-temperature coefficients, pyroelectric coefficients and coefficients of heat conduction. In the above, the body force and extrinsic bulk charge are taken to be zero. Unless other stated, the repeated indices imply summation.

Substituting eqns (2.1), (2.2), (2.4) and (2.5) into (2.6) and (2.7), eqn (2.3) into (2.8), one obtains

$$(c_{ijrs}u_r + e_{sji}\varphi)_{,si} - \beta_{ij}T_{,i} = 0 \tag{2.10}$$

$$(-\varepsilon_{is}\varphi + e_{irs}u_r)_{,si} + \tau_i T_{,i} = 0 \tag{2.11}$$

$$\lambda_{ij}T_{,ij} = 0 \tag{2.12}$$

which are the five equations to determine the five unknowns $\mathbf{u}$, φ, and T.

Given a ceramic sample, the mechanical boundary conditions are taken to be

$$\begin{aligned} \sigma_{ij}n_j &= t_i^0, \quad \text{on } S_\sigma \\ u_i &= u_i^0, \quad \text{on } S_u \end{aligned} \tag{2.13}$$

where the superscript 0 indicates that the value is prescribed. The electric boundary conditions are

$$n_i D_i = -\omega^0 \quad \text{on } S_\omega$$
$$\varphi = \varphi^0 \quad \text{on } S_\varphi \tag{2.14}$$

The thermal boundary conditions are

$$n_i h_i = h^0 \quad \text{on } S_h$$
$$T = T^0 \quad \text{on } S_T \tag{2.15}$$

In the above, the stress and induction in the environment are assumed to be negligible. The normal **n** directs towards the environment, and $S_\sigma + S_u = S_\omega + S_\varphi = S_h + S_T = S$. For a multiple-connective domain, the solution should satisfy the single-valued requirement:

$$\oint_C \mathrm{d}u_i = 0 \tag{2.16}$$

where C is an arbitrary closed contour.

In this paper our attention will only be focused on two-dimensional problems in the (x_1, x_2)-plane, i.e. nothing varies with x_3.

From eqn (2.12), one can obtain the temperature distribution (see e.g. Clements, 1973; Atkinson *et al.*, 1977; Hwu 1984, 1992)

$$T = 2\,\mathrm{Re}\,\{g'(z_t)\}\,; \quad z_t = x_1 + p_t x_2 \tag{2.17}$$

$$h_i = -2\,\mathrm{Re}\,\{(\lambda_{i1} + p_t \lambda_{i2}) g''(z_t)\} \tag{2.18}$$

where z_t is the complex variable only for temperature field and p_t satisfies the equation

$$\lambda_{22} p^2 + (\lambda_{12} + \lambda_{21}) p + \lambda_{11} = 0 \tag{2.19}$$

Since the quadratic from $\lambda_{\alpha\beta} h_\alpha h_\beta$ is positive–definite, the two roots of eqn (2.19) are complex conjugate. We denote the roots by p_t and $\bar{p}_t$, and p_t with positive imaginary part.

The general solution to eqn (2.10) and (2.11) may be obtained by letting

$$\mathbf{U} = \begin{Bmatrix} \mathbf{u} \\ \varphi \end{Bmatrix} = \mathbf{a} f(x_1 + p x_2) \tag{2.20}$$

The number p and the column $\mathbf{U}$ are determined by substituting eqn (2.20) into eqns (2.10) and (2.11), which gives

$$\mathbf{D}(p)\mathbf{a} = \{\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2 \mathbf{W}\}\mathbf{a} = 0 \tag{2.21}$$

where the superscript T indicates the transpose, and 4×4 matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{W}$ are given by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_0 & \mathbf{e}_{11} \\ \mathbf{e}_{11}^T & -\varepsilon_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}_0 & \mathbf{e}_{21} \\ \mathbf{e}_{12}^T & -\varepsilon_{12} \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} \mathbf{W}_0 & \mathbf{e}_{22} \\ \mathbf{e}_{22}^T & -\varepsilon_{22} \end{bmatrix} \tag{2.22}$$

with

$$(Q_0)_{ik} = c_{i1k1}, \quad (R_0)_{ik} = c_{i1k2}, \quad (W_0)_{ik} = c_{i2k2} \quad (i, k = 1, 2, 3)$$
$$\mathbf{e}_{ik} = \{e_{i1k} \quad e_{i2k} \quad e_{i3k}\}^T, \quad (i, k = 1, 2) \tag{2.23}$$

Matrices $\mathbf{Q}$ and $\mathbf{W}$ are symmetric and positive definite due to that the strain energy is positive. It can be proved that eqn (2.21) admits no real root and its eight roots from four conjugate pairs. Without the loss of generality, let p_1, p_2, p_3 and p_4 be the roots with positive imaginary part, and

$$z_\alpha = x_1 + p_\alpha x_2 \quad (\alpha = 1, 2, 3, 4) \tag{2.24}$$

It is convenient to express the stresses in terms of the stress function $\mathbf{\Phi}$, i.e.

$$\sigma_{i1} = -\mathbf{\Phi}_{i,2}, \quad \sigma_{i2} = -\mathbf{\Phi}_{i,1}, \quad D_1 = -\mathbf{\Phi}_{4,2}, \quad D_2 = -\mathbf{\Phi}_{4,1} \quad (i = 1, 2, 3) \tag{2.25}$$

From eqns (2.20), (2.1) and (2.2), we obtain

$$\mathbf{\Phi} = \mathbf{b}f(x_1 + px_2) \tag{2.26}$$

where

$$\mathbf{b} = (\mathbf{R}^T + p\mathbf{W})\mathbf{a} = -\frac{1}{p}(\mathbf{Q} + p\mathbf{R})\mathbf{a} \tag{2.27}$$

Equation (2.27) can be rewritten in the following standard eigenvalue problem (see e.g. Ting, 1986)

$$\mathbf{N}\xi = p\xi \tag{2.28}$$

in which

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix} \quad \xi = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}$$

$$\mathbf{N}_1 = -\mathbf{W}^{-1}\mathbf{R}^T, \quad \mathbf{N}_2 = \mathbf{W}^{-1} = \mathbf{N}_2^T, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{W}^{-1}\mathbf{R}^T - \mathbf{Q} = \mathbf{N}_3^T \tag{2.29}$$

when p_α ($\alpha = 1, 2, 3, 4$) and p_t are distinct, the general solutions for the piezothermoelastic problems are (see e.g. Lekhnitskii, 1963; Stroh, 1958)

$$\mathbf{U} = 2\,\mathrm{Re}\left\{\sum_{\alpha=1}^{4} \mathbf{a}_\alpha f_\alpha(z_\alpha) + \mathbf{c}g(z_t)\right\} \tag{2.30}$$

$$\mathbf{\Phi} = 2\,\mathrm{Re}\left\{\sum_{\alpha=1}^{4} \mathbf{b}_\alpha f_\alpha(z_\alpha) + \mathbf{d}g(z_t)\right\} \tag{2.31}$$

where

$$\mathbf{d} = (\mathbf{R}^T + p_t\mathbf{W})\mathbf{c} - \boldsymbol{\beta}_2 = -\frac{1}{p_t}(\mathbf{Q} + p_t\mathbf{R})\mathbf{c} + \frac{1}{p_t}\boldsymbol{\beta}_1 \tag{2.32}$$

$$\boldsymbol{\beta}_1 = [\beta_{11} \quad \beta_{21} \quad \beta_{31} \quad \tau_1]^T; \quad \boldsymbol{\beta}_2 = [\beta_{12} \quad \beta_{22} \quad \beta_{32} \quad \tau_2]^T \tag{2.33}$$

Substituting eqn (2.30) into eqns (2.10) and (2.11) and use of (2.21) leads to

$$\mathbf{D}(p_t)\mathbf{c} = \boldsymbol{\beta}_1 + p_t\boldsymbol{\beta}_2 \tag{2.34}$$

From this equation, we can obtain vectors $\mathbf{c}$ and $\mathbf{d}$ which are called the associated general eigenvectors of heat eigenvalue p_t. By means of eqn (2.32), eqn (2.34) can be rewritten as

$$\mathbf{N}\boldsymbol{\eta} = p_t\boldsymbol{\eta} + \mathbf{q}$$

$$\boldsymbol{\eta} = \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix}, \quad \mathbf{q} = -\begin{bmatrix} \mathbf{0} & \mathbf{N}_2 \\ \mathbf{I} & \mathbf{N}_1^T \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix} \tag{2.35}$$

Define 4×4 matrices

$$\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4]; \quad \mathbf{B} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4]$$
$$\mathbf{f}(z_\alpha) = [f_1(z_1) \quad f_2(z_2) \quad f_3(z_3) \quad f_4(z_4)] \tag{2.36}$$

As shown by Suo (1990) the analyticity of a function is not affected by the arguments z_α or z_t. The solution form appropriate for the method of analytic continuation (see e.g. Muskhelishvilli, 1954) is written as

$$T = g'(z) + \overline{g'(z)}, \quad h_i = -(\lambda_{i1} + p_t\lambda_{i2})g''(z) - (\lambda_{i1} + \bar{p}_t\lambda_{i2})\overline{g''(z)} \tag{2.37}$$

$$\mathbf{U} = \mathbf{A}\mathbf{f}(z) + \mathbf{c}g(z)\bar{\mathbf{A}}\overline{\mathbf{f}(z)} + \bar{\mathbf{c}}\overline{g(z)}$$
$$\boldsymbol{\Phi} = \mathbf{B}\mathbf{f}(z) + \mathbf{d}g(z) + \bar{\mathbf{B}}\overline{\mathbf{f}(z)} + \bar{\mathbf{d}}\overline{g(z)} \tag{2.38}$$

Note that the arguments of $g(z)$ and each component function of $\mathbf{f}(z)$ are written as $z = x_1 + px_2$ without referring to the associated eigenvalues p_t or p_α. Once the solutions of $g(z)$ and $\mathbf{f}(z)$ are obtained for a given boundary value problem, a replacement of z_t, z_1, z_2, z_3, z_4 should be made for each function to calculate field quantities. Both $\mathbf{A}$ and $\mathbf{B}$ are non-singular, so we can define

$$\mathbf{Y} = i\mathbf{A}\mathbf{B}^{-1} \tag{2.39}$$

and a bimaterial matrix is defined as

$$\mathbf{H} = \mathbf{Y}_{\text{I}} + \bar{\mathbf{Y}}_{\text{II}} \tag{2.40}$$

in which the subscript I and II are used to denote the quantities pertaining to the materials I and II, respectively.

It is easy to see that the basic equation is the combination of the problems of the piezoelectricity (see e.g. Kuo and Barnett, 1991; Suo *et al.*, 1992) and anisotropic thermoelasticity (see e.g. Hwu, 1992), but in piezothermoelasticity the equations are more complex.

3. PIEZOTHERMOELASTIC COLLINEAR INTERFACE CRACKS

Consider an arbitrary number of collinear cracks lying along the interface of two dissimilar piezothermoelastic materials, the materials I and II are located on $x_2 > 0(S_{\text{I}})$ and $x_2 < 0(S_{\text{II}})$, respectively. The materials are assumed to be perfectly bonded at all points of

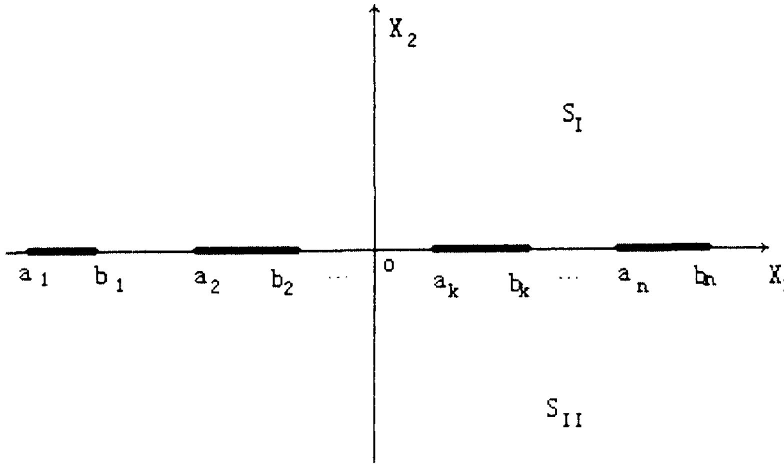

Fig. 1. Collinear interface cracks configuration.

interface $x_2 = 0$ except those lying in the region of cracks L (see Fig. 1), which are defined by the intervals

$$a_j \leqslant x_1 \leqslant b_j, \quad j = 1, 2, , \ldots, n$$

with

$$-\infty < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < \infty$$

The displacements, electric potential, traction, electric induction, temperature and heat flux are continuous across the bonded segment; the displacements, traction, electric induction and normal heat flux are also continuous on the crack faces. So

$$\mathbf{U}_{\rm I} = \mathbf{U}_{\rm II}, \quad (\mathbf{\Phi}_{,1})_{\rm I} = (\mathbf{\Phi}_{,1})_{\rm II}, \quad T_{\rm I} = T_{\rm II}, \quad (h_2)_{\rm I} = (h_2)_{\rm II} \quad x_1 \notin L \tag{3.1}$$

$$(\mathbf{\Phi}_{,1})_{\rm I} = (\mathbf{\Phi}_{,1})_{\rm II} = \mathbf{t}^0, \quad (h_2)_{\rm I} = (h_2)_{\rm II} = h^0 \quad x_1 \in L \tag{3.2}$$

in which $\mathbf{t}^0 = (t_1^0 \quad t_2^0 \quad t_3^0 \quad -\omega^0)^T$, combining eqn (3.1) and (3.2), we have

$$(\mathbf{\Phi}_{,1})_{\rm I} = (\mathbf{\Phi}_{,1})_{\rm II}, \quad (h_2)_{\rm I} = (h_2)_{\rm II} \quad -\infty < x_1 < \infty \tag{3.3}$$

Analogy to Hwu (1992), we introduce

$$\lambda = \frac{1}{2i}\lambda_{22}(p_t - \bar{p}_t) \tag{3.4}$$

$$\theta'(z) = \begin{cases} \left(1 + \dfrac{\lambda_{\rm I}}{\lambda_{\rm II}}\right) g_{\rm I}'(z), & x_2 > 0 \\ \left(1 + \dfrac{\lambda_{\rm II}}{\lambda_{\rm I}}\right) g_{\rm II}'(z), & x_2 < 0 \end{cases} \tag{3.5}$$

$$\Psi'(z) = \begin{cases} \mathbf{B}_{\rm I}\mathbf{f}'_{\rm I}(z) + \left(\frac{\lambda_{\rm II}}{\lambda_{\rm I}+\lambda_{\rm II}}\mathbf{d}_{\rm I} + \mathbf{e}_{\rm I}\right)\theta'(z), & x_2 > 0 \\ \mathbf{H}^{-1}\bar{\mathbf{H}}\left\{\mathbf{B}_{\rm II}\mathbf{f}'_{\rm II}(z) + \left(\frac{\lambda_{\rm I}}{\lambda_{\rm I}+\lambda_{\rm II}}\mathbf{d}_{\rm II} + \mathbf{e}_{\rm II}\right)\theta'(z)\right\}, & x_2 < 0 \end{cases} \tag{3.6}$$

where

$$\mathbf{e}_{\rm I} = \frac{\lambda_{\rm II}}{\lambda_{\rm I}+\lambda_{\rm II}}\left\{\mathbf{H}^{-1}\left[i\left(\mathbf{c}_{\rm I} + \frac{\lambda_{\rm I}}{\lambda_{\rm II}}\bar{\mathbf{c}}_{\rm II}\right) + \mathbf{Y}_{\rm II}\left(\mathbf{d}_{\rm I} + \frac{\lambda_{\rm I}}{\lambda_{\rm II}}\bar{\mathbf{d}}_{\rm II}\right)\right] - \mathbf{d}_{\rm I}\right\} = -\bar{\mathbf{e}}_{\rm II}$$

$$\mathbf{e}_{\rm II} = \frac{\lambda_{\rm I}}{\lambda_{\rm I}+\lambda_{\rm II}}\left\{\bar{\mathbf{H}}^{-1}\left[i\left(\mathbf{c}_{\rm II} + \frac{\lambda_{\rm II}}{\lambda_{\rm I}}\bar{\mathbf{c}}_{\rm I}\right) + \bar{\mathbf{Y}}_{\rm I}\left(\mathbf{d}_{\rm II} + \frac{\lambda_{\rm II}}{\lambda_{\rm I}}\bar{\mathbf{d}}_{\rm I}\right)\right] - \mathbf{d}_{\rm II}\right\} = -\bar{\mathbf{e}}_{\rm I} \tag{3.7}$$

Thus the problem leads to the following *Hilbert* problem

$$\theta''(x_1^-) + \theta''(x_1^-) = i\frac{\lambda_{\rm I}+\lambda_{\rm II}}{\lambda_{\rm I}\lambda_{\rm II}}h^0(x_1), \quad x \in L \tag{3.8}$$

$$\Psi'(x_1^+) + \bar{\mathbf{H}}^{-1}\mathbf{H}\Psi'(x_1^-) = \mathbf{t}^0(x_1) + \theta'(x_1^+)\mathbf{e}_{\rm I} + \theta'(x_1^-)\mathbf{e}_{\rm II}, \quad x_1 \in L \tag{3.9}$$

The solution of which is

$$\theta''(z) = \frac{1}{2\pi}\frac{\lambda_{\rm I}+\lambda_{\rm II}}{\lambda_{\rm I}\lambda_{\rm II}}\chi_0(z)\int_L \frac{h^0(x_1)\,{\rm d}x_1}{\chi_0^+(x_1)(x_1-z)} + \chi_0(z)P_{n-1}(z) \tag{3.10}$$

$$\Psi'(z) = \frac{1}{2\pi i}\mathbf{X}_0(z)\int_L \frac{1}{(x_1-z)}\{[\mathbf{X}_0^+(x_1)]^{-1}[\mathbf{t}^0(x_1) + \theta'(x_1^+)\mathbf{e}_{\rm I} + \theta'(x_1^-)\mathbf{e}_{\rm II}]\}\,{\rm d}x_1 + \mathbf{X}_0(z)\mathbf{P}_{n-1}(z) \tag{3.11}$$

where $P_{n-1}(z)$ and $\mathbf{P}_{n-1}(z)$ are arbitrary polynomials with the degree not higher than $n-1$, $x_0(z)$ and $\mathbf{X}_0(z)$ are the *Plemelj* functions defined as

$$\chi_0(z) = \prod_{j+1}^{n}(z-a_j)^{-1/2}(z-b_j)^{-1/2} \quad \mathbf{X}_0(z) = \Lambda\Gamma(z) \tag{3.12}$$

$$\Lambda = [\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4]; \quad \Gamma(z) = {\rm diag}\left[\prod_{j=1}^{n}(z-a_j)^{-\frac{1}{2}-i\varepsilon_\alpha}(z-b_j)^{-\frac{1}{2}+i\varepsilon_\alpha}\right] \tag{3.13}$$

where diag[ ] denotes a diagonal matrix, ε and λ are the eigenvalues and eigenvectors of

$$(\mathbf{H} - e^{-2\pi\varepsilon}\bar{\mathbf{H}})\lambda = 0 \tag{3.14}$$

Let $\mathbf{H} = \mathbf{M} + i\mathbf{V}$, where $\mathbf{M}$ is real and symmetric and $\mathbf{V}$ is real and antisymmetric. The results can therefore be expressed as

$$\varepsilon_{1,2} = \pm\frac{1}{\pi}\tanh^{-1}\left[(\xi^2-\zeta)^{1/2}-\xi\right]^{1/2}; \quad \varepsilon_{3,4} = \mp\frac{i}{\pi}\tan^{-1}\left[(\xi^2-\zeta)^{1/2}+\xi\right]^{1/2}$$

$$\xi = \frac{1}{4}{\rm tr}\,[(\mathbf{M}^{-1}\mathbf{V})^2]; \quad \zeta = \|\mathbf{M}^{-1}\mathbf{V}\| \tag{3.15}$$

Once the solutions of $\theta''(z)$ and $\Psi'(z)$ from eqn (3.10) and (3.11) are obtained, then $g'(z)$ and $\mathbf{f}'(z)$ can also be obtained. The traction along the interface are calculated as

$$\mathbf{\Phi}_{,1} = (\mathbf{I}+\bar{\mathbf{H}}^{-1}\mathbf{H})\Psi'(x_1) - \theta'(x_1)(\mathbf{e}_{\mathrm{I}}+\mathbf{e}_{\mathrm{II}}), \quad x_1 \notin L \tag{3.16}$$

in which $\mathbf{\Phi}_{,1} = [\sigma_{12} \quad \sigma_{22} \quad \sigma_{32} \quad D_2]^T$. If neglecting the rigid body motion, the general crack opening displacements $\Delta\mathbf{U}$ can also be calculated as

$$\Delta\mathbf{U} = \mathbf{U}_{\mathrm{I}}(x_1^+) - \mathbf{U}_{\mathrm{II}}(x_1^-) = -i\mathbf{H}[\Psi'(x_1^+) - \Psi'(x_1^-)], \quad x_1 \in L \tag{3.17}$$

In the end of this section, we discuss an asymptotic problem. Let a semi-infinite crack lies on the interface, and the crack faces are free. In this case the Hilbert problem (3.8) becomes homogeneous

$$\theta''(x_1^+) + \theta''(x_1^-) = 0, \quad x_1 \in L \tag{3.18}$$

and its solution is

$$\theta''(z) = \frac{\vartheta}{2} z^{-(1/2)}, \quad \theta'(z) = \vartheta z^{1/2} \tag{3.19}$$

where ϑ is a constant. Thus, eqn (3.9) becomes

$$\Psi'(x_1^+) + \bar{\mathbf{H}}^{-1}\mathbf{H}\Psi'(x_1^-) = \vartheta\sqrt{x_1}(\mathbf{e}_{\mathrm{I}} - \mathbf{e}_{\mathrm{II}}), \quad x_1 \in L \tag{3.20}$$

Its basic Plemelj function is

$$\mathbf{X}_0(z) = \Lambda\,\mathrm{diag}[z^{-(1/2)+i\varepsilon_\alpha}] \tag{3.21}$$

Then,

$$\Psi'(z) = \frac{1}{2\pi i}\mathbf{X}_0(z)\int_L \frac{1}{(x_1 - z)}\{[\mathbf{X}_0^+(x_1)]^{-1}[\vartheta\sqrt{x_1}(\mathbf{e}_{\mathrm{I}} - \mathbf{e}_{\mathrm{II}})]\}\,\mathrm{d}x_1 + \mathbf{X}_0(z)\mathbf{P}_{n-1}(z) \tag{3.22}$$

in which $\mathbf{P}_{n-1}(z)$ is a constant and $L \in (-\infty, 0)$.

Now employing eqn (3.16), the traction are

$$\mathbf{\Phi}_{,1} = (\mathbf{I}+\bar{\mathbf{H}}^{-1}\mathbf{H})\Psi'(x_1) - \vartheta\sqrt{x_1}(\mathbf{e}_{\mathrm{I}}+\mathbf{e}_{\mathrm{II}}), \quad x_1 \notin L \tag{3.23}$$

and the flux along the interface is

$$h_2 = \frac{-i\lambda_{\mathrm{I}}\lambda_{\mathrm{II}}}{\lambda_{\mathrm{I}}+\lambda_{\mathrm{II}}}[\theta''(x_1^+) + \theta''(x_1^-)] = \frac{-i\lambda_{\mathrm{I}}\lambda_{\mathrm{II}}}{\lambda_{\mathrm{I}}+\lambda_{\mathrm{II}}}\vartheta x_1^{-(1/2)}, \quad x_1 \notin L \tag{3.24}$$

The stress intensity factors are defined as

$$\mathbf{K} = \begin{Bmatrix} K_{\mathrm{II}} \\ K_{\mathrm{I}} \\ K_{\mathrm{III}} \\ K_D \end{Bmatrix} = \lim_{r\to 0}\sqrt{2\pi r}\Lambda\,\mathrm{diag}\,[r^{-i\varepsilon_\alpha}]\Lambda^{-1}\mathbf{\Phi}_{,1} \tag{3.25}$$

where r stands for the distance from the crack tip. The above definition is identical with the convention (see e.g. Hwu, 1993).

4. EXAMPLES

4.1. *Homogeneous media with a crack*

The simplest case is that the two media are composed of the same materials. Consider an infinite homogeneous piezothermoelastic plane containing an insulated crack. The problem is described by

$$\mathbf{A}_{\mathrm{I}} = \mathbf{A}_{\mathrm{II}} = \mathbf{A}, \quad \mathbf{B}_{\mathrm{I}} = \mathbf{B}_{\mathrm{II}} = \mathbf{B} \tag{4.1}$$

and

$$\mathbf{\Phi}_{,1} = \mathbf{t}^0, \quad h(x_1) = h^0; \quad -a \leqslant x_1 \leqslant a, \quad x_2 = 0 \tag{4.2}$$

$$\mathbf{\Phi}_{,1} = \mathbf{\Phi}_{,2} = 0, \quad h = T = 0, \quad \text{when } |z| \to \infty \tag{4.3}$$

Due to eqn (3.9) and (3.15), it can be seen that $\varepsilon = 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$. Equation (3.12) and (3.13) can be rewritten as

$$\chi_0(z) = (z^2 - a^2)^{-(1/2)}; \quad \Gamma(z) = \operatorname{diag}[(z^2 - a^2)^{-(1/2)}] \tag{4.4}$$

The solutions of the complex functions $g(z)$ and $\mathbf{f}(z)$ are obtained from eqns (3.5) and (3.6) as

$$g(z) = \frac{ih^0}{4\lambda}[z_t^2 - z_t\sqrt{z_t^2 - a^2} + a^2 \ln(z_t + \sqrt{z_t^2 - a^2})] \tag{4.5}$$

$$\mathbf{f}'(z) = \frac{h^0 i}{2\lambda}\operatorname{diag}\left[2z_\alpha - \frac{2z_\alpha^2 - a^2}{\sqrt{z_\alpha^2 - a^2}}\right]\mathbf{B}^{-1}\mathbf{e} + \frac{1}{2}\operatorname{diag}\left[1 - \frac{z_\alpha}{\sqrt{z_\alpha^2 - a^2}}\right]\mathbf{B}^{-1}\mathbf{t}^0$$
$$+ \frac{h^0 i}{2\lambda}\operatorname{diag}[\sqrt{z_\alpha^2 - a^2} - z_\alpha]\mathbf{B}^{-1}(\mathbf{d} + 2\mathbf{e}) \tag{4.6}$$

The stress $\mathbf{\Phi}_{,1}$ ahead of the crack tip along x_1-axis are

$$\mathbf{\Phi}_{,i} = \left(1 - \frac{x_1}{\sqrt{x_1^2 - \alpha^2}}\right)\mathbf{t}^0 - \frac{h^0 i}{\lambda}\frac{a^2}{\sqrt{x_1^2 - a^2}}\mathbf{e} \tag{4.7}$$

The above solution shows that the stresses are singular near the crack tip. Using eqn (3.25), the stress intensity factors are given by

$$\mathbf{K} = \lim_{x_1 \to a}\sqrt{2\pi(x_1 - a)}\mathbf{\Phi}_{,1} = -\frac{h^0 ai}{\lambda}\sqrt{\pi a}\mathbf{e} - \sqrt{\pi a}\mathbf{t}^0 \tag{4.8}$$

Similarly, the generalized crack opening displacements $\Delta\mathbf{U}$ are obtained as

$$\Delta\mathbf{U} = -\frac{2h^0}{\lambda}\frac{2x_1^2 - a^2}{\sqrt{x_1^2 - a^2}}\mathbf{He} + i\mathbf{Ht}^0\frac{x_1}{\sqrt{x_1^2 - a^2}} \tag{4.9}$$

4.2. *Bimaterials*

Consider an interface crack on $a_1 = -a$, $b_1 = a$ subjected to uniform heat flux $h = h_0$ and uniform traction $\mathbf{t} = \mathbf{t}^0$.

Just like the above section, we have

$$\theta''(z) = ih_0^*\left(1-\frac{z}{\sqrt{z^2-a^2}}\right); \quad \theta'(z) = ih_0^*(z-\sqrt{z^2-a^2}) \tag{4.10}$$

where

$$h_0^* = \frac{h_0(\lambda_{\rm I}+\lambda_{\rm II})}{2\lambda_{\rm I}\lambda_{\rm II}} \tag{4.11}$$

The Plemlj polynomial in this problem is

$$\mathbf{X}_0(z) = \Lambda\,\mathrm{diag}\,[\chi_\alpha(z)], \quad \chi_\alpha(z) = (z+a)^{-(1/2)-i\varepsilon_\alpha}(z-a)^{-(1/2)+i\varepsilon_\alpha} \tag{4.12}$$

By employing the residue theory, the boundary conditions and the single-value requirements, after some algebraic manipulation we have

$$\begin{aligned}\Psi(z) = \Lambda\Big\{&\mathrm{diag}\,[1-(z+2i\varepsilon_\alpha a)\chi_\alpha(z)]\mathbf{t}^* + ih_0^*\,\mathrm{diag}\\ &\times\left[z-\left(z^2+2i\varepsilon_\alpha az-\frac{a^2}{2}-2a^2\varepsilon_\alpha^2\right)\chi_\alpha(z)\right]\cdot(\mathbf{e}_{\rm I}^*+\mathbf{e}_{\rm II}^*)\\ &-ih_0^*\,\mathrm{diag}\,[\sqrt{z^2-a^2}-(z^2+2i\varepsilon_\alpha az-a^2-2a^2\varepsilon_\alpha^2)\chi_\alpha(z)]\\ &\times\Lambda^{-1}(\mathbf{I}-\bar{\mathbf{H}}^{1}\mathbf{H})^{-1}(\mathbf{e}_{\rm I}-\mathbf{e}_{\rm II})\Big\}\end{aligned} \tag{4.13}$$

where

$$\mathbf{t}^* = \Lambda^{-1}(\mathbf{I}+\bar{\mathbf{H}}^{-1}\mathbf{H})^{-1}\mathbf{t}^0, \quad \mathbf{e}_k^* = \Lambda^{-1}(\mathbf{I}+\bar{\mathbf{H}}^{-1}\mathbf{H})^{-1}\mathbf{e}_k \tag{4.14}$$

It is noted that the analogous calculation in Hwu (1992) is not fully correct, i.e. eqn $(31a)_3$ and (32) in that paper are error due to that the selection of branch function is not correct. This is illustrated in the Appendix.

The stress $\mathbf{\Phi}_{,1}$ ahead of the crack tip along the x_1-axis can be written as

$$\begin{aligned}\mathbf{\Phi}_{,1}(x_1) = (\mathbf{I}+\bar{\mathbf{H}}^{-1}\mathbf{H})\Lambda\Big\{&\mathrm{diag}\,[1-(x_1+2i\varepsilon_\alpha a)\chi_\alpha(x_1)]\mathbf{t}^*\\ &+ih_0^*\,\mathrm{diag}\left[x_1-\left(x_1^2+2i\varepsilon_\alpha ax_1-\frac{a^2}{2}-2a^2\varepsilon_\alpha^2\right)\chi_\alpha(x_1)\right]\cdot(\mathbf{e}_{\rm I}^*+\mathbf{e}_{\rm II}^*)\\ &-ih_0^*\,\mathrm{diag}\,[\sqrt{x_1^2-a^2}-(x_1^2+2i\varepsilon_\alpha ax_1-a^2-2a^2\varepsilon_\alpha^2)\chi_\alpha(x_1)]\\ &\cdot\Lambda^{-1}(\mathbf{I}-\bar{\mathbf{H}}^{-1}\mathbf{H})^{-1}(\mathbf{e}_{\rm I}-\mathbf{e}_{\rm II})\Big\}+ih_0^*(x_1-\sqrt{x_1^2-a^2})(\mathbf{e}_{\rm I}+\mathbf{e}_{\rm II})\end{aligned} \tag{4.15}$$

and the stress intensity factors are obtained as

$$\mathbf{K} = (\mathbf{I}+\bar{\mathbf{H}}^{-1}\mathbf{H})\Lambda\left\{\operatorname{diag}[-(1+2i\varepsilon_\alpha)\sqrt{\pi a}(2a)^{-\varepsilon_\alpha}]\mathbf{t}^*\right.$$

$$+ih_0^*\operatorname{diag}\left[-\left(\frac{1}{2}+2i\varepsilon_\alpha-2\varepsilon_\alpha^2\right)a\sqrt{\pi a}(2a)^{-i\varepsilon_\alpha}\right]\cdot(\mathbf{e}_{\mathrm{I}}^*+\mathbf{e}_{\mathrm{II}}^*)$$

$$\left.-ih_0^*\operatorname{diag}[-(2i\varepsilon_\alpha-2\varepsilon_\alpha^2)a\sqrt{\pi a}(2a)^{-\varepsilon_\alpha}]\cdot\Lambda[\mathbf{I}-\bar{\mathbf{H}}^{-1}\mathbf{H}](e_{\mathrm{I}}-\mathbf{e}_{\mathrm{II}})\right\} \tag{4.16}$$

If let $h_0^* = 0$ and multiply $\mathbf{\Lambda}^{-1}$ from left, then we get Suo's formula (see e.g. Suo, 1990).

5. A POINT HEAT SOURCE IN BIMATERIAL

In this section we develop basic singular solutions for the piezothermoelastic media induced by a point heat source with magnitude M at point $z^0(x_1^0, x_2^0)$. First, we consider the field induced by the heat source in an infinite homogeneous anisotropic medium without any crack. From eqn (2.19), we can obtain

$$p_t+\bar{p}_t = -2\lambda_{12}/\lambda_{22},\quad p_t\bar{p}_t = \lambda_{11}/\lambda_{22}$$

Then, combining eqn (2.37) and (3.4) we get

$$\begin{aligned} h_1 &= \lambda i p_t g''(z_t) - \lambda i \overline{p_t g''(z_t)} \\ h_2 &= -\lambda i g''(z_t) + \lambda i \overline{g''(z_t)} \end{aligned} \tag{5.1}$$

$$h = h_1 + ih_2 = \lambda i(p_t g'' - \bar{p}_t\bar{g}'') + \lambda(g'' - \bar{g}'') \tag{5.2}$$

since the heat source intensity is M, we have

$$M = \oint h_n\,\mathrm{d}s = \operatorname{Im}\oint h\,\mathrm{d}z \tag{5.3}$$

where the integral is around a closed contour containing the heat source. Let $g_0'(z_t) = C\ln(z_t - z_t^0)$, $g_0''(z_t) = C/(z_t - z_t^0)$, then substituting these into the above equations, it leads to

$$g_0'(z_t) = -(M/4\pi\lambda^0)\ln(z_t - z_t^0) \tag{5.4}$$

where

$$\lambda^0 = \frac{\lambda}{2}(1+\operatorname{Im}p_t)\quad z_t^0 = x_1^0 + p_t x_2^0$$

Now the interaction of a traction free interface crack with a point heat source, in Fig.

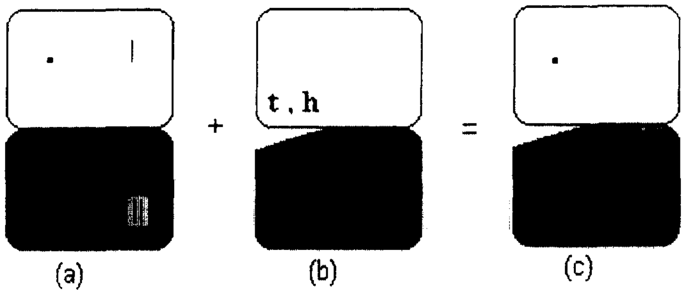

Fig. 2. (a) A point heat source in material I. (b) An interface crack with traction and heat flux prescribed on the faces. (c) Interaction between a point heat source and a interface crack.

2(c) is taken up. Without loss of generality, the heat source with intensity M is taken to be in material I, located at z^0 (x_1^0, x_2^0). This interaction problem can be solved by superposition principle illustrated in Fig. 2.

We write the solution for Fig. 2(a) in the following form

$$g''(z_t) = \begin{cases} g_1''(z_t) = g_1''(z_t) + g_0''(z_t) & x_2 > 0 \\ g_{II}''(z_t) & x_2 < 0 \end{cases} \tag{5.5}$$

where $g_1''(z_t)$ and $g_{II}''(z_t)$ are analytic in S_I and S_{II}, respectively, $g_0''(z_t)$ is expressed by eqn (5.4) and has a polar point z_t^0 in S_I.

Analogy to Suo (1990), by standard analytic continuation and from the continuity of the temperature, traction and displacements across the interface, one can obtain

$$g'(z_t) = \begin{cases} [(\lambda_I - \lambda_{II})/(\lambda_I + \lambda_{II})]\bar{g}_0'(z_t) + g_0'(z_t) & x_2 > 0 \\ 2\lambda_I/(\lambda_I + \lambda_{II})g_0'(z_t) & x_2 < 0 \end{cases} \tag{5.6}$$

$$\begin{aligned} \mathbf{f}_I'(z) = & -i\mathbf{B}_I^{-1}\mathbf{H}^{-1}\left(\frac{\lambda_I - \lambda_{II}}{\lambda_I + \lambda_{II}}\mathbf{c}_I - \frac{2\lambda_I}{\lambda_I + \lambda_{II}}\bar{\mathbf{c}}_{II} + \bar{\mathbf{c}}_I\right)\bar{g}_0'(z) \\ & -\mathbf{B}_I^{-1}\mathbf{H}^{-1}\bar{\mathbf{Y}}_{II}\left(\frac{\lambda_I - \lambda_{II}}{\lambda_I + \lambda_{II}}\mathbf{d}_I - \frac{2\lambda_I}{\lambda_I + \lambda_{II}}\bar{\mathbf{d}}_{II} + \bar{\mathbf{d}}_I\right)\bar{g}_0'(z) \\ \mathbf{f}_{II}'(z) = & -i\mathbf{B}_{II}^{-1}\mathbf{H}^{-1}\left(-\frac{\lambda_I - \lambda_{II}}{\lambda_I + \lambda_{II}}\bar{\mathbf{c}}_I + \frac{2\lambda_I}{\lambda_I + \lambda_{II}}\mathbf{c}_{II} - \mathbf{c}_I\right)g_0'(z) \\ & -\mathbf{B}_{II}^{-1}\mathbf{H}^{-1}\bar{\mathbf{Y}}_I\left(-\frac{\lambda_I - \lambda_{II}}{\lambda_I + \lambda_{II}}\bar{\mathbf{d}}_I + \frac{2\lambda_I}{\lambda_I + \lambda_{II}}\mathbf{d}_{II} - \mathbf{d}_I\right)g_0'(z) \end{aligned} \tag{5.7}$$

Notice again that z is a complex variable of the form $z = x_1 + px_2$. Thus, from eqn (2.38) the traction on the interface are

$$\begin{aligned} \mathbf{t}(x_1) &= -\mathbf{D}g_0'(\mathbf{x}_1) - \bar{\mathbf{D}}\bar{g}_0'(x_1) \\ h_2 &= -\frac{2i\lambda_I\lambda_{II}}{\lambda_I + \lambda_{II}}[g_0''(x_1) - \bar{g}_0''(x_1)] \end{aligned} \tag{5.8}$$

in which

$$\mathbf{D} = i\bar{\mathbf{H}}^{-1}\left(-\frac{\lambda_{\mathrm{I}}-\lambda_{\mathrm{II}}}{\lambda_{\mathrm{I}}+\lambda_{\mathrm{II}}}\bar{\mathbf{c}}_{\mathrm{I}}+\frac{2\lambda_{\mathrm{I}}}{\lambda_{\mathrm{I}}+\lambda_{\mathrm{II}}}\mathbf{c}_{\mathrm{II}}-\mathbf{c}_{\mathrm{I}}\right)+\bar{\mathbf{H}}^{-1}\mathbf{Y}_{\mathrm{II}}$$

$$\cdot\left(\frac{\lambda_{\mathrm{I}}-\lambda_{\mathrm{II}}}{\lambda_{\mathrm{I}}+\lambda_{\mathrm{II}}}\bar{\mathbf{d}}_{\mathrm{I}}-\frac{2\lambda_{\mathrm{I}}}{\lambda_{\mathrm{I}}+\lambda_{\mathrm{II}}}\mathbf{d}_{\mathrm{II}}+\mathbf{d}_{\mathrm{I}}\right)+\frac{\lambda_{\mathrm{I}}-\lambda_{\mathrm{II}}}{\lambda_{\mathrm{I}}+\lambda_{\mathrm{II}}}\bar{\mathbf{d}}_{\mathrm{I}}-\mathbf{d}_{\mathrm{I}}$$

$$= -2(\bar{\mathbf{e}}_{\mathrm{I}}+\bar{\mathbf{d}}_{\mathrm{I}})-\bar{\mathbf{H}}^{-1}[i(\mathbf{c}_{\mathrm{I}}+\bar{\mathbf{c}}_{\mathrm{I}})+\bar{\mathbf{Y}}_{\mathrm{I}}(\mathbf{d}_{\mathrm{I}}-\bar{\mathbf{d}}_{\mathrm{I}})] \tag{5.9}$$

Thus, the traction and heat flux prescribed on the crack faces in Fig. 2(b) can be determined as

$$\mathbf{t}^s = \mathbf{D}g_0'(x_1)+\bar{\mathbf{D}}\bar{g}_0'(x_1) \tag{5.10}$$

$$h^s = \frac{2i\lambda_{\mathrm{I}}\lambda_{\mathrm{II}}}{\lambda_{\mathrm{I}}\lambda_{\mathrm{II}}}[g_0''(x_1)-\bar{g}_0''(x_1)] \tag{5.11}$$

The solution for this problem has been examined in Section 3. In the present context, it is

$$\theta''(z) = \frac{1}{2\pi}\frac{\lambda_{\mathrm{I}}+\lambda_{\mathrm{II}}}{\lambda_{\mathrm{I}}\lambda_{\mathrm{II}}}\chi_0(z)\int_L \frac{h^0(x_1)+h^s(x_1)}{\chi_0^+(x_1)(x_1-z)}\mathrm{d}x_1+\chi_0(z)P_{n-1}(z)$$

$$\Psi'(z) = \frac{1}{2\pi i}\mathbf{X}_0(z)\int_L \frac{1}{(x_1-z)}\{[\mathbf{X}_0^+(x_1)]^{-1}[\mathbf{t}^0(x_1)+\mathbf{t}^s(x_1)+\theta'(x_1^+)\mathbf{e}_{\mathrm{I}}+\theta'(x_1^-)\mathbf{e}_{\mathrm{II}}]\}\,\mathrm{d}x_1$$

$$+\mathbf{X}_0(z)\mathbf{P}_{n-1}(z) \tag{5.12}$$

From eqn (5.12) it is seen that a point heat source produces not only temperature field, but also stress fields. As was mentioned earlier, when calculating the field quantities, z should be replaced by $z_t = x_1+p_tx_2$ and $z_\alpha = x_1+p_\alpha x_2$ ($\alpha = 1,2,3,4$), respectively, for $g(z)$ and each component of $f(z)$.

6. THE SOLUTION FOR THE INTERFACE CRACK CONTAINING A POINT HEAT SOURCE

In this section, we will discuss the interaction of a single interface crack with a point heat source. The boundary conditions on the interface are

$$(\mathbf{\Phi}_{,1})_{\mathrm{I}} = (\mathbf{\Phi}_{,1})_{\mathrm{II}} = \mathbf{t}^0,\quad (h_2)_{\mathrm{I}} = (h_2)_{\mathrm{II}} = h^0;\quad -a \leqslant x_1 \leqslant a,\quad x_2 = 0 \tag{6.1}$$

For convenience, we let $\mathbf{t}^0$ and h^0 be zero at infinite, i.e.

$$(\mathbf{\Phi}_{,1})_{\mathrm{I}} = (\mathbf{\Phi}_{,1})_{\mathrm{II}} = (\mathbf{\Phi}_{,2})_{\mathrm{I}} = (\mathbf{\Phi}_{,2})_{\mathrm{II}} = 0,\quad h = T = 0,\quad \text{when } |z| \to \infty \tag{6.2}$$

In order to obtain the full field solution, one should make out the integral in eqn (5.12).

Let

$$z_0 = x_1^0+p_tx_2^0 = \alpha+\beta i$$

$$\alpha = x_1^0+\mathrm{Re}(p_t)x_2^0,\quad \beta = \mathrm{Im}(p_t)x_2^0$$

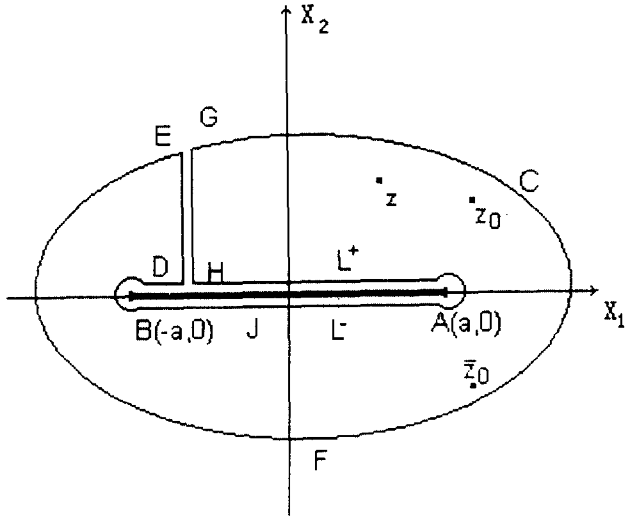

Fig. 3. The integration contour for eqn (6.4).

Then, eqn (5.18) may be written as

$$\theta''(z) = \frac{1}{2\pi}\frac{\lambda_{\rm I}+\lambda_{\rm II}}{\lambda_{\rm I}\lambda_{\rm II}}\chi_0(z)\int_L \frac{1}{\chi_0^+(x_1)(x_1-z)}\frac{2i\lambda_{\rm I}\lambda_{\rm II}}{\lambda_{\rm I}+\lambda_{\rm II}}\left(-\frac{M}{4\lambda^0\pi}\right)$$

$$\times\left(\frac{1}{x_1-z_0}-\frac{1}{x_1-\bar{z}_0}\right)\mathrm{d}x_1+\chi_0(z)P_n(z)$$

$$= -\frac{Mi}{4\lambda^0\pi^2}\chi_0(z)\int_L \frac{2\beta i}{\chi_0^+(x_1)(x_1-z)(x_1-z_0)(x_1-\bar{z}_0)}\mathrm{d}x_1+\chi_0(z)P_{n-1}(z) \quad (6.3)$$

To convert eqn (6.3) into an explicit form we start by treating the following integral

$$I = \int_C \frac{1}{\chi_0^+(x_1)(x_1-z)(x_1-z_0)(x_1-\bar{z}_0)}\mathrm{d}x_1 \quad (6.4)$$

where C denotes an integration contour depicted in Fig. 3. The integrand has three simple poles at $x_1 = z$, z_0 and $\bar{z}_0$. Thus, using the theorem of residue one obtains

$$I = 2\pi i\left\{\frac{\sqrt{z^2-a^2}}{(z-z_0)(z-\bar{z}_0)}+\frac{\sqrt{z_0^2-a^2}}{(z_0-z)(z_0-\bar{z}_0)}+\frac{\sqrt{\bar{z}_0^2-a^2}}{(\bar{z}_0-z)(\bar{z}_0-z_0)}\right\} \quad (6.5)$$

A simple analysis shows that the portion of the integral I on the contour $DEFGH$ vanishes whereas that on the contour HJD equals

$$2\int_L \frac{1}{\chi_0^+(x_1)(x_1-z)(x_1-z_0)(x_1-\bar{z}_0)}\mathrm{d}x_1$$

Consequently the final form of eqn (6.3) can be written as

$$\theta''(z) = \frac{M}{4\pi\lambda^0} 2\beta i \chi_0(z) \left\{ \frac{\sqrt{z^2-a^2}}{(z-z_0)(z-\bar{z}_0)} + \frac{\sqrt{z_0^2-a^2}}{(z_0-z)(z_0-\bar{z}_0)} + \frac{\sqrt{\bar{z}_0^2-a^2}}{(\bar{z}_0-z)(\bar{z}_0-z_0)} \right\}$$

$$+\chi_0(z)P_{n-1}(z)$$

$$= \frac{M}{4\pi\lambda^0}\chi_0(z)\left[\frac{\sqrt{z_0^2-a^2}}{(z_0-z)} - \frac{\sqrt{\bar{z}_0^2-a^2}}{(\bar{z}_0-z)}\right] + \frac{M}{2\pi\lambda^0}\frac{i\beta}{(z-z_0)(z-\bar{z}_0)} + \chi_0(z)P_{n-1}(z) \quad (6.6)$$

The infinite condition (6.2) and the single-value condition requires $P_{n-1} = 0$, thus,

$$\theta''(z) = \frac{M}{4\pi\lambda^0}\frac{1}{\sqrt{z^2-a^2}}\left[\frac{\sqrt{(\alpha-i\beta)^2-a^2}}{z-(\alpha-i\beta)} - \frac{\sqrt{(\alpha+i\beta)^2-a^2}}{z-(\alpha+i\beta)}\right] + \frac{M}{2\pi\lambda^0}\frac{i\beta}{(z-\alpha)^2+\beta^2} \quad (6.7)$$

To obtain the temperature distribution from eqn (6.7) we first integrate $\theta''(z)$ with respect to z and the result is

$$\theta'(z) = \frac{M}{4\pi\lambda^0}\left[\ln\frac{\sqrt{z^2-a^2}\sqrt{z_0^2-a^2}+(zz_0-a^2)}{\sqrt{z_0^2-a^2}} - \ln\frac{\sqrt{z^2-a^2}\sqrt{\bar{z}_0^2-a^2}+(z\bar{z}_0-a^2)}{\sqrt{\bar{z}_0^2-a^2}}\right] + C \quad (6.8)$$

The constant C in eqn (6.8) is fixed by the condition at infinite (6.2)

$$C = -\frac{M}{4\pi\lambda^0}\left\{\ln\frac{\sqrt{z_0^2-a^2}+z_0}{\sqrt{z_0^2-a^2}} - \ln\frac{\sqrt{\bar{z}_0^2-a^2}+\bar{z}_0}{\sqrt{\bar{z}_0^2-a^2}}\right\} \quad (6.9)$$

So far, we have obtained the general and exact solution for temperature distribution. Because of the complication of eqn (6.8), to write the integral eqn $(5.12)_2$ explicitly is very difficult. But, with the developing of computer techniques, the numerical integration is convenient.

As an alternative method, we can also employ series expansion to treat this problem by expanding $\mathbf{t}^s(x_1^-)$ and $\theta'(x_1^+)$ into series, in $x_1 \in L$. Thus, $\mathbf{t}^s(x_1^+)$ and $\theta(x_1^+)$ can be expressed by polynomials of x_1^n and $x_1^n\sqrt{a^2-x_1^2}$ over the internal $[-a, a]$. Using a routine procedure with the aid of Appendix, eqn $(5.12)_2$ can be determined. Since the mathematical derivation is tedious and lengthy, details of the solution are omitted here.

7. CONCLUSION

Basic equation for the thermo-electro-elastic problems and the general solution has been obtained. The case of interface cracks subjected to a point heat source has also been treated. The general solution is valid when the heat eigenvalue and the elasticity eigenvalues are distinct. For the case that they are repeated, a modified solution should be developed.

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APPENDIX: THE CONTOUR INTEGRAL OF $\mathbf{I}(z)$

Let a_1, b_1 are the end points of an arc L (Fig. A1) and on the L the Hilbert problem of vector form is given by

$$\mathbf{F}^+(s) - \mathbf{g}\mathbf{F}^-(s) = \mathbf{G}(s) \quad s \in L \tag{A1}$$

where $\mathbf{g}$ is a constant and $\mathbf{G}(s) = G(s)\mathbf{e}_k$. When solving this problem, the following integral $\mathbf{I}(z)$ is usually to be solved

$$\mathbf{I}(z) = \frac{1}{2\pi i}\int_L \frac{1}{s-z}[\mathbf{X}_0^+(s)]^{-1}\mathbf{G}^+(s)\,\mathrm{d}s = \frac{1}{2\pi i}\int_L \frac{1}{s-z}[\mathbf{X}_0^+(s)]^{-1}G^+(s)\,\mathrm{d}s\mathbf{e}_k \tag{A2}$$

where $\mathbf{X}_0(z)$ is the solution of the homogeneous equation corresponded to (A1), i.e.

$$\mathbf{X}_0^+(s) = \mathbf{g}\mathbf{X}_0^-(s) \tag{A3}$$

At first we discuss the following complex integral

$$\mathbf{I}_1(z) = \frac{1}{2\pi i}\int_C \frac{1}{s-z}[\mathbf{X}_0(s)]^{-1}G(s)\,\mathrm{d}s\mathbf{e}_k \tag{A4}$$

The closed contour C is shown in Fig. 4. Let

$$[\mathbf{X}_0(s)]^{-1}G(s) = \alpha_q s^q + \cdots + \alpha_0 + \frac{\beta_1}{s} + \cdots \quad \text{as } s \to \infty \tag{A5}$$

where $q = n+m$, n is the number of cracks, i.e. the maximum order of the expanded series of $[\mathbf{X}_0(s)]^{-1}$, m is the maximum order of the expanded series of $\mathbf{G}(s)$. According to the formula of a Cauchy type integral, we have

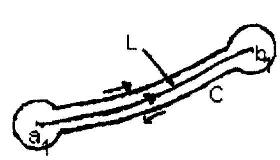

Fig. A1. The integration contour for eqn (A2).

$$\mathbf{I}_1(z) = \{[\mathbf{X}_0(z)]^{-1}\mathbf{G}(z)-\alpha_q z^q-\cdots-\alpha_0\}\cdot\mathbf{e}_k \tag{A6}$$

However, when C shrinks to L we have

$$\int_C \frac{1}{s-z}[\mathbf{X}_0(s)]^{-1}\mathbf{G}(s)\,\mathrm{d}s = \int_{a_1}^{b_1}\left[\frac{1}{s-z}[\mathbf{X}_0^+(s)]^{-1}\mathbf{G}^+(s)-\frac{1}{s-z}[\mathbf{X}_0^-(s)]^{-1}\mathbf{G}^-(s)\right]\mathrm{d}s \tag{A7}$$

According to (A3) there is $\mathbf{X}_0^-(s) = \mathbf{g}^{-1}\mathbf{X}_0^+(s)$, and if we let $G^-(s) = g^*G^+(s)$, and g^* is constant, then (A7) can be reduced to

$$\int_C \frac{1}{s-z}[\mathbf{X}_0(s)]^{-1}G(s)\,\mathrm{d}s\mathbf{e}_k = \int_{a_1}^{b_1}\frac{1}{s-z}[\mathbf{X}_0^+(s)]^{-1}(\mathbf{I}-\mathbf{g}g^*)G^-(s)\,\mathrm{d}s\mathbf{e}_k \tag{A8}$$

Finally we have

$$\frac{1}{2\pi i}\int_L \frac{G^+(s)}{s-z}[\mathbf{X}_0^+(s)]^{-1}\,\mathrm{d}s = \{[\mathbf{X}_0(z)]^{-1}G(z)-\alpha_q z^q-\cdots-\alpha_0\}(\mathbf{I}-\mathbf{g}g^*)^{-1} \tag{A9}$$

Especially when $G(s)$ is a polynomial, then $G^-(s) = G^+(s)$, i.e. $g^* = 1$, so eqn (A9) reduces to the Muskhelishivili's formula given in Muskhelishivili (1954).

Now, we return to eqn (31a)$_3$ in Hwu (1992), which is

$$\frac{1}{2\pi i}\int_{-a}^{a}\frac{i\sqrt{a^2-s^2}}{s-z}[\mathbf{X}_0^+(s)]^{-1}\,\mathrm{d}s = \{\sqrt{z^2-a^2}[\mathbf{X}_0(z)]^{-1}-\mathrm{diag}\,[z^2+2i\varepsilon_\alpha az-a^2(1+2\varepsilon_\alpha^2)]\cdot\Lambda^{-1}\}\cdot(\mathbf{I}-\mathbf{g}g^*)^{-1} \tag{A10}$$

where $\mathbf{g} = -\bar{\mathbf{H}}^{-1}\mathbf{H}$, $g^* = -1$. In Hwu (1992), g^* is taken to be 1, this is not correct. In this case, $G(s) = \sqrt{s^2-a^2}$, $G^+(s) = i\sqrt{a^2-s^2}$ and $G^-(s) = -i\sqrt{a^2-s^2}$ for $|s| < a$, so g^* should be taken to -1. Thus,

$$\mathbf{I}(z)\{\sqrt{z^2-a^2}[\mathbf{X}_0(z)]^{-1}-\mathrm{diag}\,[z(z+2i\varepsilon_\alpha a)-a^2(1+2\varepsilon_\alpha^2)]\Lambda^{-1}\}(-\bar{\mathbf{H}}^{-1}\mathbf{H})^{-1}\mathbf{e}_k$$
$$= \mathrm{diag}\left[\frac{\sqrt{z^2-a^2}}{\chi_\alpha(\alpha)}-[z(z+2i\varepsilon_\alpha a)-a^2(1+2\varepsilon_\alpha^2)]\right]\Lambda^{-1}(\mathbf{I}-\bar{\mathbf{H}}^{-1}\mathbf{H})^{-1}\mathbf{e}_k \tag{A11}$$

The above equation is the revised result of eqn (31a)$_3$ in Hwu (1992).